Canonical Transformations, Entropy and Quantization

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This paper considers various aspects of the canonical coordinate transformations in a complex phase space. The main result is given by two theorems which describe two special families of mappings between integrable Hamiltonian systems. The generating function of these transformations is determined by the entropy and a second arbitrary function which we take to be the energy function. For simple integrable systems an algebraic treatment based on the group properties of the canonical transformations is given to calculate the eigenvalue spectrum of the energy.

1. Introduction

More then 150 years ago Hamilton formulated his canonical equations. Many physicists and mathematicians have been fascinated by the symmetrical structure and the transformation properties of these equations ever since. For many years one has believed that in principle classical mechanics has already been completed, but recently, remarkable results have stimulated new intensive research in this field. There are two main directions of this research. The root of the first direction is the so-called stability problem of classical mechanics (see e.g. [1]), which is connected with the non-integrability of most of the Hamiltonian systems. The second direction of research is connected with the transition from classical mechanics to quantum mechanics. An interesting mathematical development based on group theoretical and geometrical methods is the theory of geometric quantization [2]. In this context the investigations of canonical transformations given by Moshinsky and a number of co-workers [3–6] should be mentioned.

However, between these new directions of current research there are the open questions concerning the problem of quantization of nonintegrable systems. In a recent paper [7], Kupershmidt has suggested that quantum mechanics has to do with integrable systems. Also the Bohr-Sommerfeld quantization rule underlines that the complete set of observables in quantum mechanics corresponds to the set of conserved quantities I_k (action variables) and hence to an integrable system. The aim of the present contribution is to give some hints which may be useful for the solution of this problem. The paper is organized as follows: Section 2 contains some of the basic facts about the representation of the Hamiltonian equations in a complex linear space. Some of these facts are well known especially from the harmonic oscillator (see e.g. [8]). In Sect. 3 different aspects of the canonical transformation are considered, such as infinitesimal and global linear mappings. It is very interesting to note that all global unitary transformations in the complex space are also canonical transformations. To show their practical value, a very short proof of the complete integrability of linear Hamiltonian systems is given.

Section 4 contains two special families of mappings between integrable systems. The generating function is determined by an entropy-like expression and a second arbitrary function. In Sect. 5 a special class of these mappings is considered, where the generating function is determined by entropy and energy. This is a very natural combination of the two most important quantities of physics. An application of the group properties to the harmonic oscillator model yields the correct quantum result for the energy levels. In a second example it is shown that the Fermi-transformations are connected with a nontrivial spin-structure in the case that the Hamiltonian is nonlinear in the I_k.

2. Hamiltonian Equations in Complex Form

We consider the well-known $2f$ Hamiltonian equations of motion

$$\dot{q}_k = \frac{\partial H}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H}{\partial q_k},$$
$$H = H(p_k, q_k, t), \quad k = 1, \ldots, f. \quad (2.1)$$

The p_k and q_k can be regarded as components of f-dimensional real vectors, and a complexification yields

$$z_k = q_k + i\beta p_k, \quad z = \begin{bmatrix} z_1 \\ \vdots \\ z_f \end{bmatrix} \in C^f,$$

$$z^+ = [z_1^*, \ldots, z_f^*], \qquad (2.2)$$

where z is a vector in the linear complex vector space C^f, $+$ denotes adjoint, $*$ is the complex conjugate ($z^+ = z^{T*}$) and β is a constant of the dimension area/action. A substitution $q_k = (z_k + z_k^*)/2$ and $p_k = (z_k - z_k^*)/2i\beta$ into the Hamiltonian function yields

$$H = H(z_k, z_k^*, t), \quad H = H^*. \qquad (2.3)$$

With the help of the complex partial derivative [9]

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial q_k} - \frac{i}{\beta} \cdot \frac{\partial}{\partial p_k}\right),$$

$$\frac{\partial}{\partial z_k^*} = \frac{1}{2}\left(\frac{\partial}{\partial q_k} + \frac{i}{\beta} \cdot \frac{\partial}{\partial p_k}\right)$$

one can find (2.1) to have the form

$$\dot{z}_k = -2i\beta \cdot \frac{\partial H}{\partial z_k^*}.$$

To bring this equation into a dimensionless form we perform the substitutions

$$\beta = (l_0)^2/2h, \quad z_k \to l_0 z_k,$$

$$H \to H \cdot hc/l_0, \quad t \to s\, l_0/c,$$

where c is the velocity of light, h the Planck constant, l_0 is a length, e. g. the Planck length, and s the new real parameter instead of the time. Then the equations of motion are given by

$$\frac{\mathrm{d}z_k}{\mathrm{d}s} = -i\frac{\partial H}{\partial z_k^*}, \quad H = H(z_k, z_k^*, s) = H^*. \qquad (2.4)$$

These are f ordinary complex differential equations for the vector $z(s)$ instead of the $2f$ real equations (2.1). For an arbitrary complex-valued function Φ depending on z_k, z_k^* and s it follows that

$$\frac{\mathrm{d}\Phi}{\mathrm{d}s} = \frac{\partial\Phi}{\partial s} + i\{H, \Phi\}, \qquad (2.5)$$

where the Poisson bracket is given by

$$\{H, \Phi\} \equiv \sum_{k=1}^{f}\left(\frac{\partial H}{\partial z_k} \cdot \frac{\partial \Phi}{\partial z_k^*} - \frac{\partial H}{\partial z_k^*} \cdot \frac{\partial \Phi}{\partial z_k}\right). \qquad (2.6)$$

This complex bracket has the same properties (antisymmetry, linearity, product rule and Jacobi identity) as the usual real Poisson bracket, but additionally one finds

$$\{A, B\}^* = -\{A^*, B^*\}. \qquad (2.7)$$

It is well-known that the form of the Hamiltonian equations (2.1) is is preserved under the group of the canonical coordinate transformations, and now we look for an invariance condition of (2.4). In any case, the Hamiltonian vector field or the Poisson bracket must be preserved under this transformation. We make a transformation of the following type:

$$z_k \to w_k, \quad H(z_k, z_k^*, s) \to G(w_k, w_k^*, s),$$

where

$$w_k = w_k(z_j, z_j^*, s) \quad \text{and} \quad w_k^* = w_k^*(z_j, z_j^*, s),$$

and the new coordinates are not necessarily holomorphic in the z_k. The Hamiltonian vector field is preserved under this transformation if and only if the following conditions are fulfilled:

$$\{w_j, w_k\} = 0, \quad \{w_j, w_k^*\} = \delta_{jk}, \qquad (2.8)$$

This can be easily proved by means of an elementary calculation. By splitting up these conditions into real and imaginary parts one can see that (2.8) is equivalent to the usual conditions for canonical transformations in a real phase space.

In general, H depends on $2f + 1$ real variables, the real and imaginary parts or the absolute values and phase angles of the complex coordinates z_k, respectively, and on the real parameter s. If it is possible to find a canonical transformation $z_k \to w_k$ to new variables such that the transformed Hamiltonian G is a function of $f + 1$ real variables s and I_k only, this is the usual transformation to action-angle variables (normal coordinates):

$$z_k \to w_k: H \to G(I_k, s), \quad I_k = I_k(w_k, w_k^*) = I_k^*.$$

Then the I_k are f constants of the motion, and these constants are in involution.

$$\frac{\mathrm{d}I_k}{\mathrm{d}s} = 0, \quad \{I_k, I_j\} = 0, \quad \{H, I_k\} = 0. \qquad (2.13)$$

Some examples of action variables are:

$$I_k = (w_k + w_k^*)/2, \quad I_k = (w_k - w_k^*)/2i \quad \text{or}$$

$$I_k = w_k^* w_k.$$

Especially the last case is very clear. The absolute values of the complex coordinates w_k are constants with respect to the parameter s and therefore the motion is constrained to lie on f-dimensional tori $T^f = S^1 \times S^1 \times \ldots \times S^1$.

Such a Hamiltonian system is known to be integrable, however the question of finding action-angle variables is a global one, and in most cases such normal coordinates do not exist in the whole phase space.

3. Canonical Transformations in Complex Vector Space

At the beginning of this section we want to introduce the concept of an infinitesimal canonical transformation. Such a coordinate transformation is of the form

$$z \to w: \quad w_k = z_k + \varepsilon\, \theta_k(z_j, z_j^*, s)\,, \tag{3.1}$$

where θ_k are arbitrary complex-valued functions and ε is a small parameter. This transformation is called an infinitesimal canonical transformation of order ε since it is canonical to order ε^2. We consider therefore the conditions (2.8) with the new coordinates given by (3.1):

$$\{w_j, w_k\} = \varepsilon \cdot \left(\frac{\partial \theta_k}{\partial z_j^*} - \frac{\partial \theta_j}{\partial z_k^*} \right) + O(\varepsilon^2) \overset{!}{=} O(\varepsilon^2)\,,$$

$$\{w_j, w_k^*\} = \delta_{jk} + \varepsilon \cdot \left(\frac{\partial \theta_k^*}{\partial z_j^*} + \frac{\partial \theta_j}{\partial z_k} \right)$$
$$+ O(\varepsilon^2) \overset{!}{=} \delta_{jk} + O(\varepsilon^2)\,.$$

The first condition is fulfilled by a potential ansatz

$$\theta_k = \frac{\partial \Phi}{\partial z_k^*}\,, \quad \Phi = \Phi(z_j, z_j^*, s)\,,$$

where Φ is an arbitrary complex function. Inserting this ansatz into the second condition we find

$$\frac{\partial^2 \Phi^*}{\partial z_j^* \partial z_k} + \frac{\partial^2 \Phi}{\partial z_k \partial z_j^*} = 0\,,$$

and the solution is given by

$$\Phi = \sum_{k=1}^{f} S_k(z_k, z_k^*, s) + i\Omega(z_j, z_j^*, s)\,,$$

where Ω is an arbitrary real function depending on z_k, z_k^*, s, and S_k is a real solution of the 2-dimensional Laplace-equation

$$\frac{\partial^2 S_k}{\partial z_k^* \partial z_k} = 0\,.$$

Consequently, the mapping (3.1) is an infinitesimal canonical transformation if it has the structure

$$w_k = z_k + \varepsilon \frac{\partial S_k}{\partial z_k^*} + i\,\varepsilon \frac{\partial \Omega}{\partial z_k^*}$$

or with the help of the complex bracket

$$w_k = z_k + i\,\varepsilon\,\{z_k, \Omega - iS\}\,, \quad S = \sum_{k=1}^{f} S_k\,. \tag{3.2}$$

The evolution of the system itself given by (2.4) can be considered as an infinitesimal canonical transformation with

$$\Omega = -H\,, \quad \varepsilon = \mathrm{d}s \quad \text{and} \quad S = 0:$$

$$z_k(s + \mathrm{d}s) = z_k(s) - i\{z_k, H\}\,\mathrm{d}s\,.$$

It must be underlined that this is a special case of (3.2) because the generating function H is always real. We consider the following situation to find the meaning of the imaginary part S:

$$\Omega = 0\,, \quad S_k(z_k, z_k^*, s) = S_k(I_k)\,.$$

Then the solution of the Laplace equations is given by

$$S_k = b_k + a_k \ln(I_k)\,,$$

where a_k and b_k are constants. This transformation has the structure

$$w_k = z_k \left(1 + \frac{\varepsilon\, a_k}{I_k} \right) + O(\varepsilon^2)\,,$$

i.e. the imaginary part S generates a scale transformation. The well-known feature that canonical transformations mix coordinates and momenta is most easily understood by linear transformations, e.g. by the exchange of coordinates for momenta. Therefore let us consider the global linear mapping

$$w_k = \sum_{j=1}^{f} A_{kj} z_j + \sum_{j=1}^{f} B_{kj} z_j^*\,, \tag{3.3}$$

where $\hat{A} = (A_{jk})$ and $\hat{B} = (B_{jk})$ are complex matrices which may depend on the parameter s. The calculation of the conditions (2.8) yields

$$\{w_i, w_j\} = \sum_{k=1}^{f} (A_{ik} B_{jk} - B_{ik} A_{jk}) \overset{!}{=} 0\,,$$

$$\{w_i, w_j^*\} = \sum_{k=1}^{f} (A_{ik} A_{jk}^* - B_{ik} B_{jk}^*) \overset{!}{=} \delta_{ij}\,.$$

From this follows:

Remark 1: The general linear transformation (3.3) is a canonical one if the transformation matrices fulfil the following conditions:

$$\hat{A}\hat{B}^T - \hat{B}\hat{A}^T = 0, \quad \hat{A}\hat{A}^+ - \hat{B}\hat{B}^+ = 1,$$

(T ... transpose, + ... adjoint).

Remark 2:
Each unitary transformation $w = \hat{U}z$, $\hat{U}^+ = \hat{U}^{-1}$ is also a canonical transformation.

Remark 3:
Each anti-unitary transformation $w = \hat{T}z^*$, $\hat{T}^+ = -\hat{T}^{-1}$ is also a canonical transformation.

Especially the unitary transformation group (remark 2) is of interest because it leaves the hermitian form

$$J = z_1^* z_1 + z_2^* z_2 + \ldots + z_f^* z_f \qquad (3.4)$$

invariant in the f-dimensional complex space [8]. This property is important if the Hamiltonian depends on the variables J and s. Then the equations of motion (2.4) are preserved under unitary transformations and, moreover, the Hamiltonian itself is an invariant, and the system has a symmetry. A very simple example is the 3-dimensional isotropic oscillator where the Hamiltonian itself is an invariant of the group $U(3)$.

That Hamiltonian systems are connected with symmetries is known from [10−11].

In the last part of this section we consider the case that H itself can be represented by a hermitian form

$$H = z^+ \hat{H}z, \quad \hat{H} = \hat{H}^+.$$

The equations of motion (2.4) then become linear in this special case. The eigenvalues of $\hat{H}$ which form the complete system of invariants, are preserved under unitary transformations

$$\hat{H}' = \hat{U}\hat{H}\hat{U}^+, \quad \hat{U}^+ = \hat{U}^{-1}.$$

We consider the canonical transformation (remark 2)

$$W = \hat{U}z, \quad z = \hat{U}^+ w;$$

then the Hamiltonian function is given by

$$H = z^+ \hat{H}z = w^+ \hat{U}\hat{H}\hat{U}^+ w = w^+ \hat{H}' w.$$

By a well-directed choice for $\hat{U}$ the new matrix $\hat{H}'$ has a diagonal structure and the Hamiltonian func-

tion is now of the form

$$H = \sum_{k=1}^{f} \sigma_k I_k, \quad I_k = w_k^* w_k,$$

where σ_k are real coefficients. This is the normal form for linear autonomous systems, and the corresponding equations of motion are completely integrable. The complete integrability of time-independent linear systems is a well known mathematical fact [12].

4. Canonical Equivalence of Integrable Systems

Now we consider the following problem. An integrable system with a Hamiltonian of the form

$$H = H(I_1, \ldots, I_f), \quad I_k = z_k^* z_k$$

is given. A general canonical transformation will destroy this functional structure of H:

$$H(I_1 \ldots I_f) \overset{c.t.}{\to} \tilde{H}(\tilde{I}_1 \ldots \tilde{I}_k, \tilde{z}_{k+1}, \tilde{z}_{k+1}^*, \ldots)$$

$$\overset{c.t.}{\to} \tilde{\tilde{H}}(\tilde{\tilde{z}}_k, \tilde{\tilde{z}}_k^*).$$

On the other hand the infinitesimal transformation (3.2) shows that there exist mappings between integrable systems, e. g. in the form of a scale transformation where the scale function depends on the I_k.

Instead of a general proof we consider two special families of mappings which are connected with a scale transformation. A general proof of the canonical equivalence of integrable systems is given by Gzyl in [13]. We now state

Theorem 1:

The complex transformation $w_k = (z_k)^{n_k} \cdot \exp \theta_k(I_j)$ is a canonical one if the following conditions are fulfilled:

(i) $\theta_k = n_k \cdot \dfrac{\partial}{\partial I_k}(S + i\Omega), \quad S, \Omega$ real,

(ii) $\Omega = \Omega(I_k)$ is an arbitrary real function,

(iii) $S = S(I_j) = \sum_{k=1}^{f} S_k(I_k)$, where

$$S_k = \frac{1}{2n_k}(a_k + I_k)\ln(a_k + I_k)$$
$$- \frac{I_k}{2n_k}(1 - n_k + \ln(n_k I_k^{n_k})) + b_k$$

and a_k and b_k are free parameters.

Theorem 2:

The complex transformation $w_k = (z_k^*)^{n_k} \cdot \exp \theta_k (I_j)$ is a canonical one if the following conditions are fulfilled:

(i) $\theta_k = n_k \dfrac{\partial}{\partial I_k} (S + i\Omega)$, S, Ω real,

(ii) $\Omega = \Omega (I_j)$ is an arbitrary real function,

(iii) $S = S(I_j) = \displaystyle\sum_{k=1}^{f} S_k(I_k)$, where

$$S_k = \frac{-1}{2n_k} (a_k - I_k) \ln (a_k - I_k) - \frac{I_k}{2n_k} (1 - n_k + \ln (n_k \cdot I_k^{n_k})) + b_k$$

and a_k and b_k are free parameters.

Comment: For $n_k \neq 1$ these transformations are nonbijective mappings because the inverse is not unique.

Proof: We only prove Theorem 2 because the proof of theorem 1 is analogous.

The ansatz $w_k = (z_k^*)^{n_k} \cdot \exp \theta_k (z_k, z_k^*)$ in (2.8) yields

$$\{w_j, w_k\} = w_j w_k \cdot \left(\{\theta_j, \theta_k\} + \frac{n_k}{z_k^*} \frac{\partial \theta_j}{\partial z_k} - \frac{n_j}{z_j^*} \frac{\partial \theta_k}{\partial z_j} \right),$$

$$\{w_j, w_k^*\} = w_j w_k^* \left(\{\theta_j, \theta_k^*\} - \frac{n_j n_k}{z_j^* z_k} \delta_{jk} - \frac{n_j}{z_j^*} \cdot \frac{\partial \theta_k^*}{\partial z_j} - \frac{n_k}{z_k} \cdot \frac{\partial \theta_j}{\partial z_k^*} \right). \quad (4.1)$$

The canonical condition (2.8) and the restriction $\theta_k = \theta_k (I_j)$ yield
for all j, k:

$$n_k \frac{\partial \theta_j}{\partial I_k} - n_j \frac{\partial \theta_k}{\partial I_j} = 0,$$

for $j \neq k$:

$$n_j \frac{\partial \theta_k^*}{\partial I_j} + n_k \frac{\partial \theta_j}{\partial I_k} = 0, \quad (4.2)$$

for $j = k$:

$$\exp - (\theta_k + \theta_k^*) + I_k^{n_k} \left(\frac{n_k^2}{I_k} + n_k \frac{\partial \theta_k^*}{\partial I_k} + n_k \frac{\partial \theta_k}{\partial I_k} \right) = 0.$$

The first condition of (4.2) is fulfilled by

$$\theta_k (I_j) = n_k \cdot \frac{\partial}{\partial I_k} (S + i\Omega),$$

where S and Ω are two arbitrary real functions depending on I_k. This is the proof for the statement (i) in theorem 2. By inserting this result into the second condition of (4.2) we find

$$\frac{\partial^2 S}{\partial I_k \partial I_j} = 0 \quad \text{where} \quad k \neq j$$

with the solution

$$S = \sum_{k=1}^{f} S_k (I_k).$$

By inserting this function into the last equation of (4.2) one finds

$$\exp \left(-2 n_k \frac{dS_k}{dI_k} \right) + (I_k)^{n_k} \cdot (n_k)^2 \cdot \left(\frac{1}{I_k} + 2 \frac{d^2 S_k}{dI_k^2} \right) = 0.$$

These are f ordinary differential equations of second order. A solution is possible by elementary methods and yields the special functions S_k given in Theorem 2 (iii). This completes the proof.

An explicit calculation of the absolute values of the transformed coordinates yields:

Theorem 1: $\tilde{I}_k = (I_k + a_k)/n_k$, (4.3)

Theorem 2: $\tilde{I}_k = (a_k - I_k)/n_k$, (4.4)

where $\tilde{I}_k = w_k^* w_k$ and $I_k = z_k^* z_k$, i.e. the corresponding transformations are mappings between integrable systems. Especially with $f = 1$ and $H = I$ these transformations map an oscillator of the energy E to that with a shifted energy $\tilde{E}$. An invariant with respect to the transformations (4.3) and (4.4) is the ratio of two differences

$$\text{inv} = \frac{\tilde{I}_{k1} - \tilde{I}_{k2}}{\tilde{I}_{k3} - \tilde{I}_{k4}} = \frac{I_{k1} - I_{k2}}{I_{k3} - I_{k4}},$$

which is important for the process of physical measurement [14].

It must be emphasized that in both cases the generating function S has an entropy-like structure. Moreover, we consider the special case $n_k = 1$ and $b_k = -0.5 \, a_k \ln a_k$. Then one obtains from Theorem 1

$$S = \frac{1}{2} \cdot \sum_{k=1}^{f} a_k ((1 + N_k) \ln (1 + N_k) - N_k \ln N_k) \equiv S_B/2, \quad (4.5)$$

where $N_k = I_k/a_k$ and from Theorem 2 with $n_k = 1$

and $b_k = 0.5 \cdot a_k \ln a_k$

$$S = \frac{-1}{2} \cdot \sum_{k=1}^{f} a_k ((1 - N_k) \ln (1 - N_k)$$

$$+ N_k \ln N_k) \equiv S_{\mathrm{F}}/2 . \qquad (4.6)$$

Equation (4.5) is a Bose-like and (4.6) a Fermi-like expression for the entropy if N_k is interpreted as a mean occupation number [15]. The Bose/Fermi-like property of the transformations becomes clearer if the group property of the canonical transformations is considered. In the Bose case (Theorem 1 with $n_k = 1$) we find a linear chain structure:

$$w \xrightarrow{T_{\mathrm{B}}} \tilde{w} \xrightarrow{T_{\mathrm{B}}} \tilde{\tilde{w}} \xrightarrow{T_{\mathrm{B}}} \dots ,$$

$$w* \xrightarrow{T_{\mathrm{B}}} \tilde{w}* \xrightarrow{T_{\mathrm{B}}} \tilde{\tilde{w}}* \xrightarrow{T_{\mathrm{B}}} \dots ,$$

where T_{B} stands for such a transformation. On the other hand, the Fermi-like transformations ($n_k = 1$) form a cross-chain structure:

$$w \xrightarrow{T_{\mathrm{F}}} \tilde{w} \xrightarrow{T_{\mathrm{F}}} \tilde{\tilde{w}} \dots ,$$
$$w* \xrightarrow{T_{\mathrm{F}}} \tilde{w}* \xrightarrow{T_{\mathrm{F}}} \tilde{\tilde{w}}* \dots .$$

By a combination of both types we find the possibility that a Bose-like transformation can be represented by two Fermi-like mappings with different parameters:

$$w \xrightarrow{T_{\mathrm{F}}} \tilde{w}* \xrightarrow{T_{\mathrm{F}}} \tilde{\tilde{w}} , \quad w \xrightarrow{T_{\mathrm{B}}} \tilde{\tilde{w}} .$$

This is analogous to the physical decomposition of bosons into fermions. One may speculate whether there is a real background of these mathematical structures. We think that a connection may be found by means of information theoretical considerations, i. e. by a deeper analysis of our real physical knowledge about the microsystems.

5. Canonical Transformations and Quantization

In this section we consider mappings between integrable systems which are generated by a fixed function

$$S + i \Omega = S + i H (I_k) , \qquad (5.1)$$

where $S (I_k, a_k)$ is the entropy given by (4.5) or (4.6), and $H (I_k)$ is the Hamiltonian function

(energy) of an integrable system depending on the absolute squares of z_k only.

The transformed variables $\tilde{I}_k$ then are given by

$$T_{\mathrm{B}}: \quad \tilde{I}_k = w_k^* w_k = I_k \cdot \exp \frac{\partial S_{\mathrm{B}}}{\partial I_k} = I_k + a_k$$

or by an n-fold application ($n = 0, 1, 2, \dots$)

$$(T_{\mathrm{B}})^n I_k = I_k + n a_k . \qquad (5.2)$$

On the other hand, for the Fermi-transformations

$$T_{\mathrm{F}}: \quad I_k = w_k^* w_k = I_k \cdot \exp \frac{\partial S_{\mathrm{F}}}{\partial I_k} = a_k - I_k .$$

Moreover

$$(T_{\mathrm{F}})^2 I_k = I_k , \qquad (5.3)$$

i. e. the transformation T_{F} allows only two numerical values I_k and $a_k - I_k$ for the action variables.

In the following, two examples are given in which the group properties are used to generate a discrete spectrum of measurable values. Our first example is the simple 1-dimensional harmonic oscillator with a Hamiltonian of the form

$$H = \alpha z^* z = \alpha I , \qquad (5.4)$$

where α is a constant. For the Bose-transformation one finds

$$T_{\mathrm{B}}: \quad \tilde{w} = z \sqrt{1 + a/I} \cdot \exp (i \alpha)$$

and

$$(T_{\mathrm{B}})^{-1}: \quad z = \tilde{w} \sqrt{1 - a/\tilde{I}} \cdot \exp (-i \alpha) ,$$

where $(T_{\mathrm{B}})^{-1}$ is the inverse of T_{B}. An n-fold application yields

$$(T_{\mathrm{B}})^n z = z \sqrt{1 + n a/I} \cdot \exp (i n \alpha) \stackrel{!}{=} w^{(n)} ,$$

i. e. a mapping of $w^{(n)}$ into the z-plane generates a set of points which are located on a spiral. We call these points the states and consequently z the ground state. Because all $I^{(n)}$ must be greater than zero, an application of the inverse transformation to z must reproduce the ground state. This condition fixes the numerical value $I = z^* z$ in the following way:

$$(T_{\mathrm{B}})^{-1} z \stackrel{!}{=} z \Leftrightarrow \sqrt{1 - a/I} \cdot \exp (-i \alpha) = 1 .$$

A finite solution for I is given by $I = a/2$ and $\alpha = \pi/2$. Using this result one obtains for the states

$$w^{(n)} = z \sqrt{1 + 2n} \cdot \exp (i n \pi/2) , \quad n = 0, 1, 2, \dots \quad (5.5)$$

and for the transformed energies

$$E^{(n)} = H(I^{(n)}) = \frac{\pi}{2} a (n + 1/2) . \qquad (5.6)$$

These are the correct energy levels for the quantum mechanical oscillator. It must be underlined that (5.6) contains only one free constant a, i.e. the entropy parameter fixes the energy scale. In the z-plane all states with an even/odd n lie on straight lines which are perpendicular to each other.

For this model the Fermi-transformation yields $(T_F)^2 z = z$, i.e. the transformation is isomorphic to the cyclic group C_2. But with $I = a/2$ the two energies have the same numerical value, and this is a trivial case.

The generalization to an f degrees of freedom isotropic oscillator is easy to do and also yields the correct quantum result for $E^{(n)}$.

However, the harmonic oscillator is a very simple model and especially the Hamiltonians which are nonlinear functions in the I_k are of interest. Now let us consider the Hamiltonian of the rotator,

$$H = \alpha (L_3)^2 ,$$

where L_3 is the third component of the angular momentum and α is a constant. The substitution $x_k = (w_k + w_k^*)/2$, $p_k = (w_k - w_k^*)/2 i$ yields for L_3 the representation

$$L_3 = \frac{1}{2} \cdot w^+ \hat{L}_3 w , \quad \hat{L}_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} .$$

The canonical transformation (remark 2)

$$z = \hat{U} w , \quad \hat{U} \hat{U}^+ = 1 , \quad \hat{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

diagonalizes the operator $\hat{L}_3$ and we obtain $L_3 = (z_2^* z_2 - z_1^* z_1)/2$, i.e. the Hamiltonian becomes

$$H = \alpha (I_2 - I_1)^2/4 . \qquad (5.7)$$

The corresponding Fermi-transformations are given by

$$T_F : \quad \tilde{w}_1 = z_1^* \sqrt{\frac{a_1}{I_1} - 1} \cdot \exp(i \alpha (I_1 - I_2)/2) ,$$

$$w_2 = z_2^* \sqrt{\frac{a_2}{I_2} - 1} \cdot \exp(i \alpha (I_2 - I_1)/2) .$$

For the absolute squares of z_k the relation (5.3) is valid but for the states a simple calculation yields

$$(T_F)^{2k} z_1 = z_1 \cdot \exp(i k \delta) ,$$
$$(T_F)^{2k} z_2 = z_2 \cdot \exp(-i k \delta) , \qquad (5.8)$$

where k is a natural number and δ has the form

$$\delta = \alpha (a_1 - a_2 + 2 I_2 - 2 I_1)/2 . \qquad (5.9)$$

The corresponding energies are given by (5.7) and $\tilde{H} = T_F H$, $\tilde{H} = H + (a_1 - a_2) \delta/2$.

Moreover, we presume that $a_1 - a_2 \neq 0$ and $0 \leq \delta < 2\pi$. For $\delta = 0$ only one energy value $\tilde{H} = H$ exists and the Fermi-transformation is isomorphic to the cyclic group C_2 as in the harmonic oscillator model. From the condition that T_F is isomorphic to the cyclic group C_4 a nontrivial spin structure arises. Then one obtains

$$(T_F)^4 z_k = z_k \Leftrightarrow \delta = \pi$$

and for the energy difference

$$\Delta E = \pi (a_1 - a_2)/2 . \qquad (5.10)$$

Also in this example the entropy parameters a_k fix the energy scale. It should be mentioned that a simple model with one degree of freedom only and with $H \sim I^2$ also yields an analogous spin structure. From the physical point of view the first integral I in such a model is given by the length of the angular momentum $I = |\vec{L}|$.

The simplest Bose-transformation for the model (5.7) has the form

$$(T_B)^n z_1 = z_1 \sqrt{1 + \frac{n a_1}{I_1}}$$
$$\cdot \exp\left(i \alpha \frac{n}{2} (I_1 - I_2 + (n - 1) (a_1 - a_2)/2)\right) ,$$

$$(T_B)^n z_2 = z_2 \sqrt{1 + \frac{n a_2}{I_2}}$$
$$\cdot \exp\left(-i \alpha \frac{n}{2} (I_1 - I_2 + (n - 1) (a_1 - a_2)/2)\right) ,$$

and the energy differences are given by

$$\Delta E_{(n)} = \tilde{H}_{(n)} - H$$
$$= \frac{\alpha}{4} (a_1 - a_2)^2 \cdot \left(n^2 + 2 n \cdot \frac{I_1 - I_2}{a_1 - a_2}\right) . \qquad (5.11)$$

We consider the case of a trivial spin structure only, i.e. $T_F \sim C_2$. Then one obtains with $\delta = 0$ and (5.9) and (5.11) for the Bose sequence

$$\Delta E_{(n)} = \frac{\alpha}{4}(a_1 - a_2)^2 n(n+1).$$

Basically, it is to be expected that the nonlinear functions $H(I_k)$ have a nontrivial spin structure and that the combination of Bose- and Fermi-transformations yields a very rich structure of the spectrum.

Let us consider in a last short remark the limit $a_k/I_k \to 0$ for the Bose-transformation. Because S_B tends to zero, the transformation degenerates to a pure phase rotation. The condition which fixes the ground state $((T_B)^{-1}z = z$ for the harmonic oscillator) drops out and the motion on the circle can be described by a continuous parameter, and this is obviously the classical motion of an integrable system with the frequencies $\partial H/\partial I_k$. From this point of view there is no place for chaotic quantum behaviour.

6. Concluding Remarks

It has been demonstrated in this paper that the complex phase space is a very helpful tool for studying the canonical transformations. Especially the integrable systems are of interest because they have a high degree of symmetry. That most classical completely integrable systems are related to Lie algebras is known from the results of mathematical research conducted in the past few years [17].

However, physics deals with measurable properties, and from this point of view our physical knowledge has always a structural or algebraic nature [14, 16]. In Hamiltonian mechanics such a natural algebraic structure is given by the canonical transformations. The most important result of our considerations is the entropy-like character of the generating functions (4.5) and (4.6). The entropy is known to be the information function. Doubtlessly, a physical information is determined by a numerical value and a unit, or rather by an observable and a measuring rule. One of the basic facts in classical physics is the assumption that the ratio of measure and observable can be arbitrarily small, and this corresponds to the limit $(a_k/I_k) \to 0$. But because of the atomistic structure of matter this is not true, and especially the measure itself is also of material nature. London refers to such a situation as the discrete manifold [14].

The examples in Sect. 5 show that the I_k are the observables, the a_k are the corresponding measures and $H(I_k)$ represents the concrete system. The canonical transformations with a fixed measure then generate all measurable states. However, there are many open problems which are connected with the choice of the full set of observables I_k and the function $H(I_k)$ for a given experiment. The author will deal with these aspects in a later paper.

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